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VARIABLE GAIN FEEDBACK CONTROL FOR LINEAR SAMPLED-DATA SYSTEMS WITH BOUNDED CONTROL*

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Abstract. A design method of a controller with a variable feedback gain is presented for a linear sampled-data plant subject to a control constraint. The performance index considered here is a quadratic function of the state. A set of the steady state LQ optimal gains and their associated linear regions, i.e., the sets of initial conditions such that the control via the gain satisfies the constraint, are utilized to determine the control signal. The resulting control law is a state feedback via the state-dependent piecewise constant feedback gain which becomes progressively higher as the state approaches the equilibrium point. The closed loop system is more effective than the steady state LQ optimal regulator which satisfies the constraint. Calculations for obtaining the control signal are relatively simple as compared with perfect optimal control.

It is also shown that the linear region can be described by a set of inequalities. A few examples of simulation experiments are also presented.

Key Words—System design, variable gain, bounded control, quadratic cost.

1. Introduction

In physical systems, all variables are generally bounded. Since the control variables are also constrained, there should be a limit to the response speed in a real control system. As for the design problem of a control system subject to a constraint on control input, there have been many works so far from various points of view (Deley and Franklin, 1965; Frankena and Sivan, 1979; Gutman and Hagander, 1985; Kalman, 1957; Kiendl, 1982; Kosut, 1983). A typical approach to this problem is to find a control function minimizing a performance index under constrained conditions. This approach usually leads to a two-point boundary value problem which yields an open loop bang-bang solution. In usual applications, however, such an optimal control scheme is not adopted due to its complexity and expensiveness.

On the other hand, there is the Linear-Quadratic (LQ) optimal regulator which minimizes a quadratic cost function of the system state and control under no constraint. In this case, the optimal control law is a linear state feedback and hence, relatively simple in its analysis and realization. The trade-off between response speed and control amplitude can be made by suitably selecting the weighting factors in the quadratic cost function. Because of the linearity of the system, the control variable becomes small as the state approaches its equilibrium point (usually the origin). A more effective method of control is to maintain

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the control variable which is close to its maximal allowable value. Use of a variable feedback gain enables us to design such a controller.

In the present paper, therefore, we have intended to develop a method for designing a controller having a variable feedback gain for a linear sampled-data plant under a constrained control condition, using the performance index represented by a quadratic function of the state. The ideas introduced lie especially in the following phases: (i) choosing a set of LQ optimal feedback gains corresponding to a sequence of increasing weights on the state in the usual quadratic cost function and, for each gain, finding the linear region, i.e., the set of initial conditions such that the control via the gain is satisfactory for the constraint, (ii) at each sampling instant, applying the highest gain whose linear region includes the current state. Such viewpoints imply that the state penalty becomes progressively higher as the state approaches the origin; that is, a lower-gain linear control is used far from the origin, while a higher-gain control is used near it.

There will be discussions organized in Sec. 2.1 as preliminaries, Sec. 2.2 presenting the design algorithm, and Sec. 3 illustrating a few examples. The theorems in Sec. 2.1 are only for the mathematical rigor of the proposed technique.

2. The design of the controller

2.1 Preliminaries The plant, the constraint and the performance index are as follows;

$$x(t+1) = Ax(t)+bu(t), \quad t = 0,1,\cdots, \tag{1}$$

$$|u(t)| \leq 1, \tag{2}$$

$$\mathrm{PI} = \sum_{t=0}^{\infty} x'(t)Qx(t), \quad Q = C'C, \tag{3}$$

where x is the n-state vector, u the scalar control variable, A an $(n \times n)$-constant matrix, b an n-constant vector, and Q an $(n \times n)$-positive semidefinite matrix. The prime denotes the transpose. It is assumed that (A, b) is controllable and (C, A) observable.

First, consider the performance index PI achieved by the steady state LQ optimal regulator and the set of initial conditions which satisfy the constraint (2). The LQ optimal regulator is designed such that the following quadratic function is minimized:

$$J = \sum_{t=0}^{\infty} [\rho x'(t)Qx(t)+u(t)^2], \quad \rho>0. \tag{4}$$

The resulting optimal control law is the state feedback expressed by

$$u(t) = -k'(\rho)x(t), \tag{5}$$

where the feedback gain $k(\rho)$ is obtained by solving the steady state Riccati equation. The closed loop system is represented by

$$x(t+1) = \hat{A}(\rho)x(t), \tag{6}$$

where

$$\hat{A}(\varrho) \triangleq A - bk'(\varrho). \tag{7}$$

The performance index PI of the system (6) can be evaluated by

$$\text{PI} = x'(0)P(\varrho)x(0), \tag{8}$$

where the cost matrix $P(\varrho)$ is the solution of the Lyapunov equation, i.e.,

$$P(\varrho) = \hat{A}'(\varrho)P(\varrho)\hat{A}(\varrho) + Q. \tag{9}$$

The set of all initial conditions that satisfy the constraint (2) is defined by

$$\Gamma(\varrho) \triangleq \{x_0 \,|\, |u(t)| = |k'(\varrho)\hat{A}(\varrho)^t x_0| \le 1,\ t = 0, 1, \cdots\}. \tag{10}$$

The set $\Gamma(\varrho)$ is called the linear region for the gain $k(\varrho)$. The relation (8) is true, if and only if the initial condition $x(0)$ belongs to $\Gamma(\varrho)$. The performance index PI is monotonically nonincreasing with the increasing weighting factor ϱ. Moreover, if there is the relation $\varrho_1 < \varrho_2$, the following inequality,

$$P(\varrho_2) \le P(\varrho_1), \tag{11}$$

holds. This fact seems to be natural and, in practice, the inequality (11) can be easily seen. A proof is given in Appendix. By increasing ϱ, however, the linear region becomes narrow, because the norm of the feedback gain $k(\varrho)$ generally increases.

We proceed to present the design of the control system having a variable feedback gain. As already mentioned above, there are the following ideas: A set of LQ optimal feedback gains corresponding to different weighting factors in the quadratic function J is chosen, and then, at each sampling instant, the highest gain satisfying the constraint on the control variable is applied. Theorem 1 contains the fundamental rules for this scheme, Theorems 2 and 3 provide a method for constructing the linear region, and algorithm suggests how to design the controller.

Let ϱ be a positive function of the sampling instant t,

$$\varrho = \varrho(t) > 0. \tag{12}$$

The corresponding LQ optimal gain and the cost matrix are denoted by $k(\varrho(t))$ and $P(\varrho(t))$, respectively.

Theorem 1. If $\{\varrho(t)\}$ is a nondecreasing sequence, i.e.,

$$\varrho(t+1) \ge \varrho(t), \quad t = 0, 1, \cdots, \tag{13}$$

the control law expressed by

$$u(t) = -k'(\varrho(t))x(t) \tag{14}$$

gives a smaller PI as compared with the constant gain feedback,

$$u(t) = -k'(\varrho(0))x(t). \tag{15}$$

The constraint (2) is also satisfied, if $\varrho(t)$ is chosen so as to hold the condition,

$$x(t) \in \Gamma(\varrho(t)). \tag{16}$$

Proof. The closed loop system given by (1) and (14) is expressed by

$$x(t+1) = \hat{A}(\varrho(t))x(t). \tag{17}$$

It is easily seen that the performance index of the system (17) is evaluated by

$$\text{PI} = x'(0)P(\varrho(0))x(0) + \sum_{t=1}^{\infty} x'(t)[P(\varrho(t)) - P(\varrho(t-1))]x(t). \tag{18}$$

Since $\varrho(t)$ is nondecreasing, from (11) and (18), the inequality

$$\text{PI} \leq x'(0)P(\varrho(0))x(0) \tag{19}$$

holds. This proves the first half of the theorem. The latter half is obvious from the definition of the set $\Gamma(\varrho)$. Besides, if the condition (16) is satisfied at each sampling instant, the inequality (19) is true, and, therefore, the stability of the system (17) is assured.

It is obvious that a constrained control cannot bring the state to the origin from an arbitrary initial state, if the matrix A is unstable. In such an unstable case, the linear region is also bounded for any $\varrho > 0$, and the control law can be applied only to the linear region.

Next, we consider the construction method of the linear region $\Gamma(\varrho)$ defined by (10). From the definition of $\Gamma(\varrho)$, a state x is a member of $\Gamma(\varrho)$, if the inequality,

$$|u(t)| = |k'(\varrho)A(\varrho)^t x| \leq 1 \tag{20}$$

holds for arbitrary nonnegative integer t. However, from Theorem 2, it can be said that the condition $x \in \Gamma(\varrho)$ can be confirmed by checking (20) for a finite set of t, i.e., $t = 0 \sim j_0$, where the number j_0 is determined depending on the gain $k(\varrho)$.

Define the following n-vectors,

$$z(t) \triangleq \hat{A}'(\varrho)^t k(\varrho), \quad t = 0, 1, \cdots, \tag{21}$$

and the sets

$$L(t) \triangleq \{x_0 \mid |u(t)| = |z'(t)x_0| \leq 1\}, \tag{22}$$

$$S(j) \triangleq \bigcap_{t=0}^{j} L(t). \tag{23}$$

It is evident that $S(j)$ is a monotonically nonincreasing set and its limit is $\Gamma(\varrho)$. Since $L(t)$ is a set bounded by two parallel hypersurfaces, $\Gamma(\varrho)$ is a polyhedral convex set. Let $\tilde{S}(j)$ be the convex combination of $\pm z(t)$ $(t=0\sim j)$, i.e.,

$$\begin{aligned}\tilde{S}(j) &= C\{z(t),\ \ t = 0\sim j\} \\ &\triangleq \{z \mid z = \sum_{t=0}^{j} \alpha(t) z(t),\ \ \sum_{t=0}^{j} |\alpha(t)| \le 1\},\end{aligned} \tag{24}$$

where $C\{z(t),\ t=0\sim j\}$ means the convex combination of $\pm z(t)$, $t=0\sim j$. Since

$$\begin{aligned}S(j) &= \{x \mid |z'x| \le 1,\ z \in \tilde{S}(j)\},\\ \tilde{S}(j) &= \{z \mid |x'z| \le 1,\ x \in S(j)\},\end{aligned}$$

$\tilde{S}(j)$ and $S(j)$ are called the dual region of $S(j)$ and $\tilde{S}(j)$, respectively. The region $\tilde{S}(j)$ is also a monotonically nondecreasing set and the dual region of its limit is $\Gamma(\varrho)$.

Theorem 2. The linear region $\Gamma(\varrho)$ can be described by

$$\Gamma(\varrho) = S(j_0), \tag{25}$$

where

$$j_0 = \min\{j \mid z(j+1) \in \tilde{S}(j)\}. \tag{26}$$

Proof. Operating $\hat{A}'(\varrho)$ to $^{\forall} z \in \tilde{S}(j)$ yields

$$\hat{A}'(\varrho)\tilde{S}(j) = C\{z(t),\ \ t = 1\sim(j+1)\}. \tag{27}$$

Since there is also the relation,

$$\tilde{S}(j+1) = C\{z(t),\ \ t = 0\sim(j+1)\}, \tag{28}$$

$\tilde{S}(j+1)$ is represented by the convex combination of $\tilde{S}(j)$ and $\hat{A}'(\varrho)\tilde{S}(j)$. Thus, if the relation $\hat{A}'(\varrho)\tilde{S}(j) \subset \tilde{S}(j)$, i.e.,

$$z(j+1) \in \tilde{S}(j) \tag{29}$$

holds, $\tilde{S}(j)$ arrives at its limit, and $\Gamma(\varrho)$ can be expressed by (25). It has also been shown by the above discussion that $\pm z(t)$ $(t=0\sim j_0)$ are the extreme points of $\tilde{S}(j_0)$. That is, the set $\{z(t)\}_{t=0}^{j_0}$ is necessary and sufficient to describe $\Gamma(\varrho)$.

In order to find j_0, the condition (29) needs to be examined for each j. An efficient method for examining (29) is presented in the following:

Define the $(n \times 2(j+1))$-matrix D by

$$D \triangleq [\hat{z}_1, \hat{z}_2, \cdots, \hat{z}_{2(j+1)}], \tag{30}$$

where

$$\begin{aligned}\hat{z}_i &= z(i-1) - z(j+1),\\ \hat{z}_{i+j+1} &= -z(i-1) - z(j+1),\quad i = 1\sim(j+1).\end{aligned}$$

Theorem 3. The following three statements are equivalent:

i) The vector $z(j+1)$ is not a member of $\tilde{S}(j)$, i.e., $z(j+1) \notin \tilde{S}(j)$.

ii) The inequality $h'D<0$ (all the elements of $h'D$ are negative) holds for some $h \in R^n$.

iii) The linear equation $Dy = 0$, $y \in R^{2(j+1)}$, having the condition $y \geq 0$ (all the elements of y are nonnegative), has no solution except for $y = 0$.

Proof. Statement ii) means that there exists a hyperplane H which satisfies the conditions $z(j+1) \in H$ and $H \cap \tilde{S}(j) = \emptyset$. Since $\tilde{S}(j)$ is a convex set, statement ii) is equivalent to statement i). Assume that statement ii) holds, but statement iii) does not; i.e., there exists a vector $y \geq 0$ $(y \neq 0)$ that satisfies the equation $Dy = 0$. Then, $h'Dy \neq 0$, since $h'D<0$ and $y \geq 0$ $(y \neq 0)$. On the other hand, there is the relation $h'Dy = 0$, since $Dy = 0$. Accordingly, the assumption is violated. Statement iii) indicates that the zero vector does not belong to the convex combination of $\hat{z}_1 \sim \hat{z}_{2(j+1)}$. This implies statement i); i.e., statement ii). Hence, statement iii) is equivalent to statement ii).

It is easily seen that statement iii) can be examined by solving the following:
Linear programing: Minimize the function, i.e.,

$$f = c'y, \quad c' = (1, 1, \cdots, 1)' \in R^{2(j+1)}, \tag{31}$$

subject to the conditions,

$$Dy = 0, \tag{32}$$

$$y \geq 0. \tag{33}$$

Considering (32) and (33), and the fact that all elements of c are positive, the optimal value of f, i.e., f^*, is 0 or ∞. If $f^* = 0$, statement iii) holds. If $f^* = \infty$, statement iii) does not hold. Linear programing can be solved efficiently by the well-known simplex method.

2.2 Design algorithm If the LQ optimal gain which corresponds to the largest $\varrho(t)$, satisfying the condition (16), is applied at any sampling instant, the control having the control variable which is close to its maximal allowable value may be realized. Since an on-line search for the appropriate $\varrho(t)$ is difficult, a simplified technique is proposed as follows;

Consider an increasing sequence of the weighting factors $\{\hat{\varrho}_i\}_{i=0}^{N}$, i.e.,

$$0<\hat{\varrho}_0<\hat{\varrho}_1< \cdots <\hat{\varrho}_N.$$

As it is expected from (11) and (18) that a larger $\hat{\varrho}_i$ makes the performance index smaller, the largest $\hat{\varrho}_i$ that satisfies the condition (16) is to be chosen from the sequence $\{\hat{\varrho}_i\}_{i=0}^{N}$ at each sampling instant.

The weighting factor $\hat{\varrho}_0$ should be determined so that the linear region $\Gamma(\hat{\varrho}_0)$ may contain all initial conditions of interest. As already mentioned, the linear region cannot be unbounded, if the plant is unstable, and initial states which do not belong to any linear region are out of consideration. The factor $\hat{\varrho}_N$ should be selected so that the region $\Gamma(\hat{\varrho}_N)$ covers small disturbances or system noises. The number N may be decided in consideration of the trade-off between the computational efficiency and the performance index. A larger N gives smaller performance index, but takes more CPU time to calculate the control signal. The factor $\hat{\varrho}_i$ $(i = 1 \sim (N-1))$ is chosen, taking into account the size of the linear region $\Gamma(\hat{\varrho}_i)$. For instance, if the factors $\hat{\varrho}_0$ and $\hat{\varrho}_N$, and the number N are determined, there are factors such that the ratios of the norm between two adjacent gains are constant, i.e.,

$$\|k(\hat{\varrho}_i)\| / \|k(\hat{\varrho}_{i-1})\| = [\|k(\hat{\varrho}_N)\| / \|k(\hat{\varrho}_0)\|]^{\frac{1}{N}}, \quad i = 1 \sim N.$$

Algorithm. The algorithm is divided into off-line and on-line portions.
Off-line steps:
Step 1. Considering the above mentioned matters, determine the nubmer N, the sequence $\{\hat{\varrho}_i\}_{i=0}^{N}$ and the LQ optimal gain $k(\hat{\varrho}_i)$. The gain $k(\hat{\varrho}_i)$ is obtained by solving the Riccati equation, i.e.,

$$k(\hat{\varrho}_i) = \frac{1}{1+b'R(\hat{\varrho}_i)b} A'R(\hat{\varrho}_i)b, \quad i = 0 \sim N, \tag{34}$$

where $R(\hat{\varrho}_i) > 0$ is the solution of the Riccati equation,

$$X = A'XA + \hat{\varrho}_i Q - \frac{1}{1+b'Xb} A'Xbb'XA. \tag{35}$$

Step 2. Construct the linear region $\Gamma(\hat{\varrho}_i)$, i.e., calculate the vector,

$$z_i(\tau) = \hat{A}'(\hat{\varrho}_i)^{\tau} k(\hat{\varrho}_i), \quad \tau = 0 \sim j_0^{\,i}, \quad i = 0 \sim N, \tag{36}$$

where the number $j_0^{\,i}$ satisfies (26). These vectors are utilized to examine the condition (16).

On-line steps:
Step 3. Put $v = 0$ and $t = 0$.

Step 4. Sample the state $x(t)$.

Step 5. Choose the largest number v according to

$$v = \max\{i \mid x(t) \in \Gamma(\hat{\varrho}_i), \quad i = v \sim N\}. \tag{37}$$

The condition $x(t) \in \Gamma(\hat{\varrho}_i)$ is true, if and only if the following inequality holds:

$$|z_i'(\tau)x(t)| \leq 1, \quad \tau = 0 \sim j_0^{\,i}. \tag{38}$$

Step 6. Set up the control $u(t)$ according to

$$u(t) = -k'(\hat{\varrho}_v)x(t), \tag{39}$$

put $t = t+1$, and return to Step 4.

While the on-line steps always provide a nondecreasing sequence of the factor $\hat{\varrho}_v$, there are disturbances in real environments. Therefore, the on-line steps need to allow the factor $\hat{\varrho}_v$ to be reduced when a disturbance occurs. In practice, we shall use the following equation in place of (37):

$$v = \max\{i \mid x(t) \in \Gamma(\hat{\varrho}_i), \quad i = 0 \sim N\}. \tag{40}$$

3. Examples

We consider the following second order plant and performance index:

$$x(t+1) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{41}$$

$$\mathrm{PI} = \sum_{t=0}^{\infty} x_1(t)^2, \tag{42}$$

where x_1 denotes the first element of the state x. The eigenvalues of the matrix A are both equal to one. Assume that the initial state $x(0)$ is a member of the set,

$$\Omega = \{x \mid |x_1| \leq 1, \quad |x_2| \leq 1\}. \tag{43}$$

Such a weighting factor as $\varrho_0 = 0.01$ can be selected so as to hold the following relation:

$$\Omega \subset \Gamma(\hat{\varrho}_0). \tag{44}$$

An examination of the relation (44) can be replaced by checking whether the extreme points of Ω, namely, the vectors $(1,1)'$ and $(1,-1)$ belong to $\Gamma(\hat{\varrho}_0)$. Letting the factor $\hat{\varrho}_N \to \infty$, the corresponding steady state LQ optimal regulator becomes the minimal-time deadbeat regulator.

Consider the following three cases:

1) $N=1,\ \hat{\varrho}_0 = 0.01,\ \hat{\varrho}_1 \to \infty,$

2) $N=2,\ \hat{\varrho}_0 = 0.01,\ \hat{\varrho}_1 = 0.225,\ \hat{\varrho}_2 \to \infty,$

3) $N=4,\ \hat{\varrho}_0 = 0.01,\ \hat{\varrho}_1 = 0.048,\ \hat{\varrho}_2 = 0.225,\ \hat{\varrho}_3 = 1.64,\ \hat{\varrho}_4 \to \infty.$

The ratio of the norm $\|k(\hat{\varrho}_N)\|/\|k(\hat{\varrho}_0)\|$ is about $(1.4)^4$, and $\hat{\varrho}_i (i = 1 \sim N)$ is determined such that the ratios of the norm between two adjacent gains are about $(1.4)^4$, $(1.4)^2$ and 1.4, respectively. The LQ optimal gains $k(\hat{\varrho}_i)$ and the number j_0^i in Case 3) are shown in Table 1. As is obvious from Table 1, for example, the condition $x(t) \in \Gamma(\hat{\varrho}_4)$ can be examined from the $j_0^4 + 1 (=2)$ inequalities, i.e.,

Table 1. The LQ optimal gain $k(\hat{\varrho}_i)$ and the number j_0^i for Case 3)

i	$\hat{\varrho}_i$	$k(\varrho_i)$	j_0^i
0	0.01	$(-0.3618,\ 0.4417)'$	5
1	0.048	$(-0.4872,\ 0.6441)'$	3
2	0.225	$(-0.6292,\ 0.9181)'$	2
3	1.64	$(-0.8121,\ 1.3672)'$	1
4	∞	$(-1.0\quad ,\ 2.0\quad)'$	1

$$|z_4'(0)x(t)| \leq 1, \ |z_4'(1)x(t)| \leq 1,$$

exist, where

$$z_4(0) = k(\hat{\varrho}_4) = (-1, 2)', \ z_4(1) = \hat{A}'(\hat{\varrho}_4)k(\hat{\varrho}_4) = (0, -1)'.$$

These inequalities are easily checked numerically.

The performance indices of the three cases with respect to the initial condition, $x(0) = (1, -1)'$, are

$$\text{PI} = 46.85, \ 29.62 \text{ and } 16.84,$$

respectively. It is evident that the performance index decreases as the number N increases. Figure 1 shows the configurations of the linear regions of Case 3). Figure 2 illustrates the trajectories obtained by applying the control of Case 3), the control via the constant gain $k(\hat{\varrho}_0)$ (PI = 47.47) and the perfect optimal control (PI = 11.0), which also coincides with the time optimal control in this case. The computational time for obtaining $u(t)$ depends mostly on the calculations for finding the number v; that is, it is dependent most of all on checking the $\sum_{i=0}^{N}(j_0^i + 1)$ inequalities given by (38). In Case 3), the CPU time for each $u(t)$ is about 0.2 sec. by using the BASIC program on a 16-bit personal computer (the clock frequency of CPU is 5 MHz).

It should be noted that there are systems where some of the number j_0^i become large; e.g., the case where the system matrix $\hat{A}(\hat{\varrho}_i)$ has eigenvalues close to the boundary of the unit circle. In such cases, as shown in Yoshida, Nishimura and Yonezawa (1985), the number of inequalities can be reduced, if the linear region is approximated by some polyhedral convex set. The computational time required for this control scheme can be considered to be intermediate between the linear control case and the optimal control one presented in

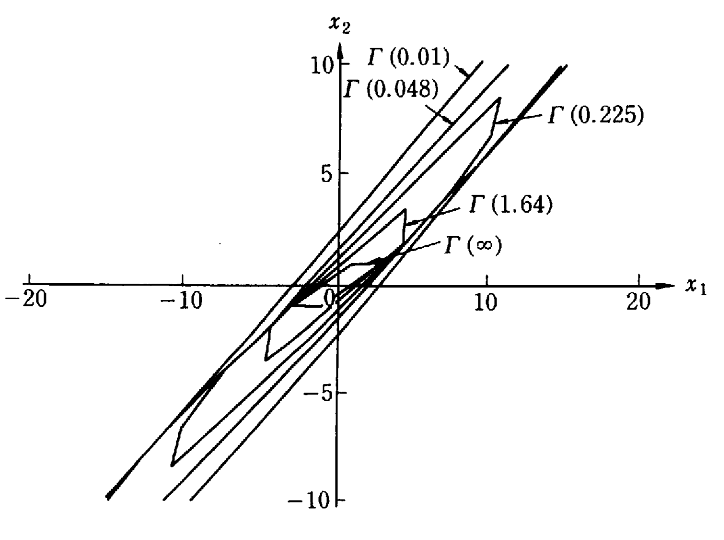

Fig. 1. The linear regions of Case 3).

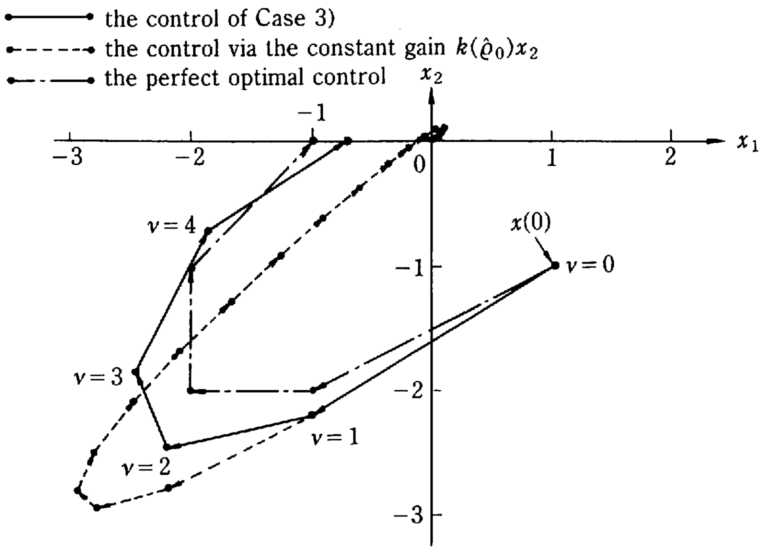

Fig. 2. The simulation results for $x(0)=(1,-1)'$.

Deley and Franklin (1965) and Kalman (1957).

4. Conclusions

A design method has been presented for synthesizing the controller having a variable gain for a linear plant under a control constraint. A set of the LQ optimal gains and their linear regions are utilized to determine the control signal which is more effective than the steady state LQ optimal regulator subject to a similar constraint on the control variable.

The proposed design algorithm is divided into the off-line steps and the on-line ones. The former contains the determination of a sequence of increasing weights in the quadratic cost function, calculations of the LQ optimal gains, and the construction of their associated linear regions which are described by a set of inequalities. The latter involves the algorithm for finding the highest gain that satisfies the control constraint by using the linear regions. This can be made mostly by evaluating the $\sum_{i=0}^{N}(j_0^i+1)$ inequalities represented by $|z'x|\leq 1$, while the trade-off between the computational time and the performance index can be done by selecting the number N.

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Appendix: Proof of the inequality (11)

The LQ optimal gain $k(\varrho)$ is given by

$$k(\varrho) = \frac{1}{1+b'R(\varrho)b} A'R(\varrho)b, \tag{A.1}$$

where $R(\varrho)>0$ satisfies the Riccati equation,

$$R(\varrho) = A'R(\varrho)A+\varrho Q - \frac{1}{1+b'R(\varrho)b} A'R(\varrho)bb'R(\varrho)A. \tag{A.2}$$

Differentiating both sides of (A.2) with respect to ϱ and taking (7) and (A.1) into account, we obtain

$$\frac{dR(\varrho)}{d\varrho} = \hat{A}'(\varrho) \frac{dR(\varrho)}{d\varrho} \hat{A}(\varrho)+Q. \tag{A.3}$$

It can be seen from (9) and (A.3) that the relation,

$$\frac{dR(\varrho)}{d\varrho} = P(\varrho), \tag{A.4}$$

holds. From (A.1) and (A.4), there is also the relation,

$$\frac{dk(\varrho)}{d\varrho} = \frac{1}{1+b'R(\varrho)b} (A'P(\varrho)b-b'P(\varrho)bk(\varrho)). \tag{A.5}$$

Substituting the relation (A.4) ~~and~~ into (A.3), differentiating both sides of (A.3) with respect to ϱ and taking (A.5) into account, we obtain

$$\frac{dP(\varrho)}{d\varrho} = \hat{A}'(\varrho) \frac{dP(\varrho)}{d\varrho} \hat{A}(\varrho) - \frac{2}{1+b'R(\varrho)b} (A'P(\varrho)b-b'P(\varrho)bk(\varrho)) \times (A'P(\varrho)b-b'P(\varrho)bk(\varrho))'. \tag{A.6}$$

Equation (A.6) is also the Lyapunov equation. Since $\hat{A}(\varrho)$ is stable and the second term of the right-hand side of (A.6) is a negative semidefinite matrix, then

$$\frac{dP(\varrho)}{d\varrho} \leq 0. \tag{A.7}$$

This implies the inequality (11).